Recurrence Factors for Normalized Associated Legendre Functions

Robert Kooima

2010

1 Introduction

The objective of this document is to compartmentalize the methods for computating normalized associated Legendre functions (ALFs): to separate out each of the constituent parts of the ALF computation distinguishing one application from another, and to describe and demonstrate each concept separately. In so doing, this work defines a set of building blocks from which a desired ALF recursion for a specific application may be constructed. These building blocks include the choice of normalization, the selection of order-wise versus degree-wise recurrence, the optional inclusion of the Condon-Shortley phase, and application of normalized ALFs to either the real or complex spherical harmonic transform.

$$\bar{P}_\ell^m(x) = q_\ell^m \, P_\ell^m(x). \tag{1}$$

[1]

$$Y_\ell^m(\theta,\phi) = \bar{P}_\ell^m(\cos\theta)\, e^{im\phi}. \tag{2}$$

colatitude $0 \le \theta \le \pi$ and longitude $0 \le \phi \le 2\pi$

ALFs satisfy the orthogonality relationship

$$\int_{-1}^{1} \bar{P}_\ell^m(x)\, \bar{P}_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}$$

where $\delta_{\ell\ell'} = 1$ when $\ell = \ell'$ or 0 otherwise. The complex spherical harmonics satisfy[2]

$$\int_S Y_\ell^{m*}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi = \frac{4\pi}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}\,\delta_{mm'}$$

[1]To resolve any confusion, the bar $\bar{P}_\ell^m$ used here denotes normalization, and the asterisk Y_ℓ^{m*} denotes complex conjugation. Some authors denote normalizion with a tilde $\tilde{P}_\ell^m$, and complex conjugation with an overline $\overline{Y_\ell^m}$.

[2]A more terse presentation might display this integral as

$$\int_\Omega Y_\ell^{m*}(\omega)\, Y_{\ell'}^{m'}(\omega)\, d\omega$$

However, the verbose presentation is chosen for consistency, as Y_ℓ^m is more clearly defined in terms of θ and ϕ.

1.1 Normalization

1.1.1 Orthonormalization

For the complex basis, this takes the form

$$q_\ell^m = \sqrt{\frac{(2\ell+1)}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}. \tag{3}$$

The orthonormal ALFs satisfy

$$\int_{-1}^{1} \bar{P}_\ell^m(x)\, \bar{P}_{\ell'}^m(x)\, dx = \frac{1}{2\pi}\,\delta_{\ell\ell'},$$

and spherical harmonic orthogonality has unit norm,

$$\int_S Y_\ell^{m*}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\,\delta_{mm'}.$$

1.1.2 Geodesy 4π normalization

The field of geodesy commonly uses what is known as the "4π normalization,"

$$q_\ell^m = \sqrt{(2\ell+1)\,\frac{(\ell-m)!}{(\ell+m)!}}. \tag{4}$$

4π-normalized ALFs satisfy the orthogonality relation

$$\int_{-1}^{1} \bar{P}_\ell^m(x)\, \bar{P}_{\ell'}^m(x)\, dx = 2\,\delta_{\ell\ell'},$$

and the complex spherical harmonics satisfy

$$\int_S Y_\ell^{m*}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi = 4\pi\,\delta_{\ell\ell'}\,\delta_{mm'},$$

from whence comes the name.

1.1.3 Schmidt seminormalization

The magnetics and quantum mechanisms communities use the Schmidt seminormalization,

$$q_\ell^m = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}. \tag{5}$$

Seminormalized ALFs satisfy the orthogonality relation

$$\int_{-1}^{1} \bar{P}_\ell^m(x)\,\bar{P}_{\ell'}^m(x)\,dx = \frac{1}{2\ell+1}\,\delta_{\ell\ell'},$$

and the spherical harmonic orthogonality satisfies

$$\int_S Y_\ell^{m*}(\theta,\phi)\,Y_{\ell'}^{m'}(\theta,\phi)\,\sin\theta\,d\theta\,d\phi = \frac{4\pi}{2\ell+1}\,\delta_{\ell\ell'}\,\delta_{mm'}.$$

1.2 Spherical Harmonics

The real form of the spherical harmonic basis uses a slightly modified normalization, introducing a factor of $\sqrt{2}$ where $m \neq 0$.

$$Y_{\ell m}(\theta,\phi) = \begin{cases} \sqrt{2}\,\bar{P}_\ell^{+m}(\cos\theta)\,\cos(m\phi) & \text{if } m > 0 \\ \bar{P}_\ell^m(\cos\theta) & \text{if } m = 0 \\ \sqrt{2}\,\bar{P}_\ell^{-m}(\cos\theta)\,\sin(-m\phi) & \text{if } m < 0 \end{cases} \quad (6)$$

2 ℓ-varying ALF Recurrences

We begin with the ℓ-varying recursive definition of the associated Legendre functions and their first derivative. On the diagonal, where degree ℓ equals order m, the associated Legendre functions are

$$P_\ell^\ell(x) = (2\ell-1)!!\,(1-x^2)^{\ell/2}. \quad (7)$$

Moving one step up in degree,

$$P_{\ell+1}^\ell(x) = x\,(2\ell+1)\,P_\ell^\ell(x). \quad (8)$$

Elsewhere, the ALFs fit the following three-term recurrence [1] (8.5.3), [3] (2.5.20),

$$(\ell-m)P_\ell^m(x) = x(2\ell-1)P_{\ell-1}^m(x) - (\ell+m-1)P_{\ell-2}^m(x) \quad (9)$$

See Figure 1 for a graphical depiction of the relationship between these three equations.

ALF derivatives are defined in terms of these values [1] (8.5.4),

$$(x^2-1)\frac{dP_\ell^m}{dx}(x) = x\ell P_\ell^\ell(x) - (\ell-m)P_{\ell-1}^m \quad (10)$$

The derivative on the diagonal follows from the definition of the trianglular derivative, given that $P_{\ell-1}^\ell = 0$.

$$(x^2-1)\frac{dP_\ell^\ell}{dx}(x) = x\ell P_\ell^\ell(x) \quad (11)$$

In practice, (7) is not as useful as the recurrence

$$P_0^0(x) = 1$$
$$P_\ell^\ell(x) = \sqrt{1-x^2}\,(2\ell-1)\,P_{\ell-1}^{\ell-1}(x) \quad (12)$$

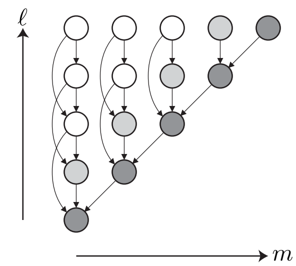

Figure 1: The dependancies of the ℓ-varying recurrence giving P_ℓ^m. Darker elements correspond to (7), lighter to (8), and white to (9).

and (8) may be omitted by stipulating $P_{\ell-1}^\ell = 0$ in (9).

In general, ℓ-varying ALF calculations take the form of a pair of recurrences, one tracing the diagonal, and another filling the triangular region above. This set of equations forms the core of the discussion of this section.

$$P_0^0(x) = q_0^0$$
$$P_\ell^\ell(x) = \sqrt{1-x^2}\,a_\ell\,P_{\ell-1}^{\ell-1}(x)$$
$$P_\ell^m(x) = x\,b_\ell^m\,P_{\ell-1}^m(x) - c_\ell^m\,P_{\ell-2}^m(x)$$
$$\frac{dP_\ell^m}{dx}(x) = \frac{x\ell}{x^2-1}\,P_\ell^m(x) - \frac{1}{x^2-1}\,d_\ell^m\,P_{\ell-1}^m(x) \quad (13)$$

where the unnormalized recurrence factors are

$$a_\ell = 2\ell-1 \qquad b_\ell^m = \frac{2\ell-1}{\ell-m}$$
$$c_\ell^m = \frac{\ell+m-1}{\ell-m} \qquad d_\ell^m = \ell+m. \quad (14)$$

2.1 ℓ-varying Normalized ALFs

Normalization factors vary from field to field following the demands of each application, but they universally include factorials of the degree ℓ and order m. For example, we will presently examine the geodetic normalization,

$$q_\ell^m = \sqrt{(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}}.$$

Given machine arithmetic, these factorials are numerically intractable for non-small ℓ and m, and thus the normalization factor as a whole cannot be separately computed. Instead, its computation must be integrated into the ALF recurrences themselves.

Such recurrences do fit the general form of (13), but the factors a_ℓ, b_ℓ^m, c_ℓ^m, and d_ℓ^m are modified. Given an arbitrary normalization q_ℓ^m, we wish to solve for these new factors $\bar{a}_\ell$, $\bar{b}_\ell^m$, $\bar{c}_\ell^m$, and $\bar{d}_\ell^m$. We begin with the generalized form of (13) with normalization coefficients inserted,

$$q_\ell^\ell P_\ell^\ell = (\dots)\,\bar{a}_\ell\, q_{\ell-1}^{\ell-1} P_{\ell-1}^{\ell-1}(x)$$

$$q_\ell^m P_\ell^m(x) = (\dots)\,\bar{b}_\ell^m q_{\ell-1}^m P_{\ell-1}^m(x) - \bar{c}_\ell^m q_{\ell-2}^m(x) P_{\ell-2}^m(x)$$

$$q_\ell^m \frac{dP_\ell^m}{dx}(x) = (\dots)\, q_\ell^m P_\ell^m(x) - (\dots)\,\bar{d}_\ell^m\, q_{\ell-1}^m P_{\ell-1}^m(x)$$

Intuitively, the P_ℓ^m values on the right hand side of each equation arrive already normalized, with their normalization coefficients q_ℓ^m attached. We must undo these normalizations and apply the unnormalized recurrence factors (14). Ultimately, we'll also need to apply the normalization coefficient q_ℓ^m on the left hand side, and it is advantageous to include it in each of the recurrence terms immediately. The first term of the derivative has the same normalization coefficient as the result, so it requires no special consideration. Arithmetically,

$$\bar{a}_\ell = (2\ell-1)\frac{q_\ell^\ell}{q_{\ell-1}^{\ell-1}}, \tag{15}$$

$$\bar{b}_\ell^m = \left(\frac{2\ell-1}{\ell-m}\right)\frac{q_\ell^m}{q_{\ell-1}^m}, \tag{16}$$

$$\bar{c}_\ell^m = \left(\frac{\ell+m-1}{\ell-m}\right)\frac{q_\ell^m}{q_{\ell-2}^m}, \tag{17}$$

$$\bar{d}_\ell^m = (\ell+m)\frac{q_\ell^m}{q_{\ell-1}^m}. \tag{18}$$

These evade numerical intractibility because the factorials in the numerators cancel those in the denominators. We'll see this happen in detail in the next section.

2.2 Orthonormalization

We will now derive the four factors for the ℓ-varying ALF recurrence for the complex form of the orthonormalization. This presentation strives for clarity and completeness, as it establishes a general pattern for the derivation of recurrence factors applicable to any ℓ-varying normalization. This derivation is the first of several, and only this first one is so verbose.

2.2.1 Base case

The base case of the recurrence follows straightforwardly from the evaluation of (3) for $\ell = m = 0$.

$$\bar{P}_0^0(x) = \sqrt{\frac{1}{4\pi}}, \tag{19}$$

2.2.2 Diagonal recurrence

On the diagonal, where $\ell = m$, (3) reduces to

$$q_\ell^\ell = \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}.$$

Substituting this into (15),

$$\bar{a}_\ell = (2\ell-1)\frac{\sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}}{\sqrt{\frac{2\ell-1}{4\pi(2\ell-2)!}}}$$

Cancelling $1/\sqrt{4\pi}$, flipping the denominator, and concatenating the radicals,

$$\bar{a}_\ell = (2\ell-1)\sqrt{\frac{(2\ell+1)}{(2\ell)!}\frac{(2\ell-2)!}{(2\ell-1)}}.$$

Bringing like terms together,

$$\bar{a}_\ell = (2\ell-1)\sqrt{\frac{(2\ell+1)}{(2\ell-1)}\frac{(2\ell-2)!}{(2\ell)!}}.$$

Here, the factorials cancel. This is the key advantage of this formulation, allowing normalized ALFs to be computed for large ℓ.

$$\bar{a}_\ell = (2\ell-1)\sqrt{\frac{(2\ell+1)}{(2\ell-1)}\frac{1}{(2\ell)(2\ell-1)}}.$$

Finally, all occurrances of $2\ell-1$ cancel and we're left with the simplified recurrence factor, ready to be dropped into (13) to compute orthonormalized ALFs on the diagonal.

$$\bar{a}_\ell = \sqrt{\frac{2\ell+1}{2\ell}} \tag{20}$$

2.2.3 Triangle recurrence, first term

Now on to the recurrence giving the triangular region above the diagonal. Substituting (3) into (16),

$$\bar{b}_\ell^m = \frac{2\ell-1}{\ell-m}\frac{\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}}{\sqrt{\frac{(2\ell-1)}{4\pi}\frac{(\ell-m-1)!}{(\ell+m-1)!}}}.$$

Cancelling, flipping, and concatenating,

$$\bar{b}_\ell^m = \frac{2\ell-1}{\ell-m}\sqrt{\frac{(2\ell+1)}{1}\frac{(\ell-m)!}{(\ell+m)!}\frac{1}{(2\ell-1)}\frac{(\ell+m-1)!}{(\ell-m-1)!}}.$$

Bringing like terms together,

$$\bar{b}_\ell^m = \sqrt{\frac{(\ell-m)!}{(\ell-m-1)!}\frac{(\ell+m-1)!}{(\ell+m)!}\frac{(2\ell-1)}{\sqrt{2\ell-1}}\frac{\sqrt{2\ell+1}}{(\ell-m)}}.$$

Cancelling the factorials,

$$\bar{b}_\ell^m = \sqrt{\frac{\ell-m}{\ell+m}\frac{\sqrt{2\ell-1}}{1}\frac{\sqrt{2\ell+1}}{(\ell-m)}}.$$

Mopping up the cancellation that arrises as a result, we arrive at the simplified recurrence factor for the first term.

$$\bar{b}_\ell^m = \sqrt{\frac{(2\ell-1)}{(\ell-m)}\frac{(2\ell+1)}{(\ell+m)}} \tag{21}$$

2.2.4 Triangle recurrence, second term

We proceed similarly with the second term of the triangular recurrence by substituting (3) into (17)

$$\bar{c}_\ell^m = \frac{\ell+m-1}{\ell-m}\frac{\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}}{\sqrt{\frac{(2\ell-3)}{4\pi}\frac{(\ell-m-2)!}{(\ell+m-2)!}}}.$$

Cancelling, flipping, and concatenating,

$$\bar{c}_\ell^m = \frac{\ell+m-1}{\ell-m}\sqrt{\frac{(2\ell+1)}{1}\frac{(\ell-m)!}{(\ell+m)!}\frac{1}{(2\ell-3)}\frac{(\ell+m-2)!}{(\ell-m-2)!}}$$

Bringing like terms together,

$$\bar{c}_\ell^m = \frac{\ell+m-1}{\ell-m}\sqrt{\frac{(2\ell+1)}{(2\ell-3)}\frac{(\ell-m)!}{(\ell-m-2)!}\frac{(\ell+m-2)!}{(\ell+m)!}}.$$

Again the factorials evaporate, though they leave behind a bit more residue than last time. Having eliminated the factorials from both terms of the triangular recursion, it becomes possible to compute ALFs for large ℓ and m

$$\bar{c}_\ell^m = \frac{\ell+m-1}{\ell-m}\sqrt{\frac{(2\ell+1)}{(2\ell-3)}\frac{(\ell-m)}{(\ell+m)}\frac{(\ell-m-1)}{(\ell+m-1)}}.$$

Bringing like terms together for another round of cancellation,

$$\bar{c}_\ell^m = \sqrt{\frac{(2\ell+1)}{(2\ell-3)}\frac{(\ell-m-1)}{(\ell+m)}\frac{\sqrt{\ell-m}}{(\ell-m)}\frac{(\ell+m-1)}{\sqrt{\ell+m-1}}}.$$

We finally arrive at the simplified recurrence factor for the second term.

$$\bar{c}_\ell^m = \sqrt{\frac{(2\ell+1)}{(2\ell-3)}\frac{(\ell+m-1)}{(\ell+m)}\frac{(\ell-m-1)}{(\ell-m)}}. \tag{22}$$

2.2.5 Derivative

The normalization factor for the derivative is derived by substituting (3) into (18).

$$\bar{d}_\ell^m = (\ell+m)\frac{\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}}{\sqrt{\frac{(2\ell-1)}{4\pi}\frac{(\ell-m-1)!}{(\ell+m-1)!}}}.$$

Cancelling, flipping, and concatenating,

$$\bar{d}_\ell^m = (\ell+m)\sqrt{\frac{(2\ell+1)}{1}\frac{(\ell-m)!}{(\ell+m)!}\frac{1}{(2\ell-1)}\frac{(\ell+m-1)!}{(\ell-m-1)!}}.$$

Eliminating the factorials,

$$\bar{d}_\ell^m = (\ell+m)\sqrt{\frac{(2\ell+1)}{(2\ell-1)}\frac{(\ell-m)}{(\ell+m)}}.$$

Canceling and reorganizing,

$$\bar{d}_\ell^m = \sqrt{(\ell^2-m^2)\frac{(2\ell+1)}{(2\ell-1)}}, \tag{23}$$

2.2.6 Summary

In summary, orthonormalized ALFs and first derivatives for the complex spherical harmonic transform may be computed with machine arithmetic by applying the ℓ-varying recurrence relation (13) with recurrence factors

$$\begin{aligned}
\bar{a}_\ell &= \sqrt{\frac{2\ell+1}{2\ell}} \\
\bar{b}_\ell^m &= \sqrt{\frac{(2\ell-1)}{(\ell-m)}\frac{(2\ell+1)}{(\ell+m)}} \\
\bar{c}_\ell^m &= \sqrt{\frac{(2\ell+1)}{(2\ell-3)}\frac{(\ell+m-1)}{(\ell+m)}\frac{(\ell-m-1)}{(\ell-m)}} \\
\bar{d}_\ell^m &= \sqrt{(\ell^2-m^2)\frac{(2\ell+1)}{(2\ell-1)}}
\end{aligned} \tag{24}$$

with $\bar{P}_0^0(x) = \sqrt{1/4\pi}$.

2.3 Geodesy 4π Normalization

The geodesy 4π normalization (4) differs from the orthonormalization (3) (perhaps counterintuitively) by the removal of the constant 4π from the denominator of the radical. The impact of this on the recurrence is slight, as this constant never appears in a numerator of (15–18) without also appearing in a denominator, cancelling out of each of the recursion factors immediately.

The one significant difference is the removal of the constant from the base case. Evaluating the 4π normalization coefficient (4) at $\ell = m = 0$,

$$\bar{P}_0^0(x) = 1. \quad (25)$$

Otherwise, the ℓ-varying 4π recurrence factors are the same as the orthornormalized factors (24). In this context, we may note that equations (20–23) are confirmed by Equations (12), (13), and (16) of Holmes and Featherstone [4]. Their base case will follow from our discussion of the real spherical harmonic transform in Section 3.4.

2.4 Seminormalization

The seminormalization (5) retains the factorial fraction in common with the ortho- and 4π normalizations, but does away with the non-constant factor $2\ell+1$. This change does not cancel out as smoothly as the orthonormalization's $1/\sqrt{4\pi}$, and the derivation of the seminormalized recurrence factors must work through the substitution of (5) into (15–18). We'll forego the detailed presentation of this derivation, as the process mirrors that of Section 2.2. The resulting factors for substitution into (13) are

$$\begin{aligned} \bar{a}_\ell &= \sqrt{\frac{2\ell-1}{2\ell}} \\ \bar{b}_\ell^m &= \frac{2\ell-1}{\sqrt{\ell^2-m^2}} \\ \bar{c}_\ell^m &= \sqrt{\frac{(\ell+m-1)}{(\ell+m)}\frac{(\ell-m-1)}{(\ell-m)}} \\ \bar{d}_\ell^m &= \frac{\ell+m}{\sqrt{(\ell+m)(\ell-m+1)}} \end{aligned} \quad (26)$$

with $\bar{P}_0^0(x) = 1$.

3 m-varying ALF Recurrences

In some circumstances it can be advantageous to organize a spherical harmonic transform implementation to work by degree instead of by order. This may be due to the mapping of the transform onto a parallel platform, or the subsetting of the active ALFs to fit within limited memory resources.

Whatever the motivation, the ALFs have an equally stable m-varying formulation based upon another three-term recurrence. The ALFs along the diagonal are computed as before, using (12), restated here for reference, with the variable substituted for clarity.

$$\begin{aligned} P_0^0(x) &= 1 \\ P_m^m(x) &= \sqrt{1-x^2}\,(2m-1)\,P_{m-1}^{m-1}(x) \end{aligned} \quad (27)$$

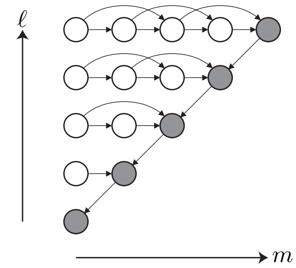

Figure 2: The dependancies of the m-varying recurrence giving P_ℓ^m. Dark elements correspond to (7) and white to (28).

We'll forego mirroring (8) and separately formulating P_m^{m-1} this time, since the definition follows trivially from the stipulation of $P_m^{m+1} = 0$ in the triangular recurrence, from Edmonds [3] (2.5.24),

$$\sqrt{1-x^2}(\ell+m+1)(\ell-m)P_\ell^m(x) = 2(m+1)xP_\ell^{m+1}(x) - \sqrt{1-x^2}P_\ell^{m+2}(x). \quad (28)$$

The m-varying derivative, from Abramowitz [1] (8.5.2),

$$\frac{dP_\ell^m}{dx}(x) = \frac{(\ell+m)(\ell-m+1)}{\sqrt{x^2-1}}P_\ell^{m-1}(x) - \frac{xm}{x^2-1}P_\ell^m(x). \quad (29)$$

See Figure 2 for a graphical depiction of the relationship between these definitions.

$$\begin{aligned} P_0^0(x) &= q_0^0 \\ P_m^m(x) &= \sqrt{1-x^2}\,a^m\,P_{m-1}^{m-1}(x) \\ P_\ell^m(x) &= \frac{x}{\sqrt{1-x^2}}\,b_\ell^m P_\ell^{m+1}(x) - c_\ell^m P_\ell^{m+2}(x) \\ \frac{dP_\ell^m}{dx}(x) &= \frac{1}{\sqrt{x^2-1}}\,d_\ell^m P_\ell^{m-1}(x) - \frac{xm}{x^2-1}P_\ell^m(x) \end{aligned} \quad (30)$$

$$a_\ell = 2\ell-1 \qquad b_\ell^m = \frac{2(m+1)}{(\ell+m+1)(\ell-m)}$$
$$c_\ell^m = \frac{1}{(\ell+m+1)(\ell-m)} \qquad d_\ell^m = (\ell+m)(\ell-m+1). \quad (31)$$

$$\bar{a}_\ell = (2\ell-1)\frac{q_\ell^\ell}{q_{\ell-1}^{\ell-1}} \quad (32)$$

$$\bar{b}_\ell^m = \frac{2(m+1)}{(\ell+m+1)(\ell-m)}\frac{q_\ell^m}{q_\ell^{m+1}} \quad (33)$$

$$\bar{c}_\ell^m = \frac{1}{(\ell+m+1)(\ell-m)} \frac{q_\ell^m}{q_\ell^{m+2}}. \quad (34)$$

$$\bar{d}_\ell^m = (\ell+m)(\ell-m+1) \frac{q_\ell^m}{q_\ell^{m-1}}. \quad (35)$$

3.1 Orthonormalization

Along the diagonal, the m-varying recurrence is the same as the ℓ-varying recurrence (20), thus the base case remains the same,

$$\bar{P}_0^0(x) = \sqrt{\frac{1}{4\pi}}. \quad (19)$$

The diagonal recursion factor also remains, though we continue to substitute m for ℓ for the sake of consistency.

$$\bar{a}^m = \sqrt{\frac{2m+1}{2m}}$$

The rest are derived by substituting (3) into (33–35).

$$\begin{aligned} \bar{b}_\ell^m &= \frac{2(m+1)}{\sqrt{(\ell+m+1)(\ell-m)}}, \\ \bar{c}_\ell^m &= \sqrt{\frac{(\ell+m+2)}{(\ell+m+1)} \frac{(\ell-m-1)}{(\ell-m)}}, \\ \bar{d}_\ell^m &= \sqrt{(\ell+m)(\ell-m+1)}. \end{aligned} \quad (36)$$

3.2 Geodesy 4π Normalization

The discussion of the ℓ-varying recurrence in Section 2.3 also applies to the m-varying recurrence. The orthonormalization (3) differs from the 4π normalization (4) only by a constant, so the m-varying 4π recurrence factors are the same as the orthonormal recurrence factors (36), and the base case eliminates the constant, $\bar{P}_0^0(x) = 1$. These results are confirmed by Holmes and Featherstone [4], given there as Equations (19) and (22).

3.3 Seminormalization

The seminormalization (5) differs from the orthonormalization by a factor $\sqrt{2\ell+1}/4\pi$. In the context of an m-varying recurrence, where ℓ is held constant, a factor of ℓ simply cancels out. Thus the derivations (33–35) of the seminormal recurrence factors $\bar{b}_\ell^m$, $\bar{c}_\ell^m$, and $\bar{d}_\ell^m$ directly resemble the derivation of the corresponding orthonormal recurrence factors, and the results are the same.

The one circumstance where ℓ varies is in the definition (32) of the diagonal recurrence factor $\bar{a}_\ell$. Of course, the computation on the diagonal is identical between the ℓ-varying and m-varying recurrences. Therefore, the seminormalized factors for the m-varying recurrence are a selection of three of the orthonormal m-varying factors (36) and one seminormal ℓ-varying factor (26).

$$\begin{aligned} \bar{a}_\ell &= \sqrt{\frac{2\ell-1}{2\ell}} \\ \bar{b}_\ell^m &= \frac{2(m+1)}{\sqrt{(\ell+m+1)(\ell-m)}}, \\ \bar{c}_\ell^m &= \sqrt{\frac{(\ell+m+2)}{(\ell+m+1)} \frac{(\ell-m-1)}{(\ell-m)}}, \\ \bar{d}_\ell^m &= \sqrt{(\ell+m)(\ell-m+1)}, \end{aligned} \quad (37)$$

with $\bar{P}_0^0(x) = 1$.

3.4 The real normalization

As given by (6), the normalization of the real form of the spherical harmonic basis introduces a factor of $\sqrt{2}$ where $m \neq 0$. The impact of this is slight. The $\sqrt{2}$ is substituted into both the numerator and the denominator of all three of recurrence factor equations (15), (16), and (17), and thus it immediately cancels out wherever it appears, leaving the general solutions (20), (21), and (22) intact.

However, a_1 is evaluated in terms of q_1^1 and q_0^0, so the $\sqrt{2}$ appears in the numerator, but not the denominator. Thus, while the complex form of q_1^1 evaluates to $\sqrt{3/2}$, the real form evaluates to $\sqrt{3}$. In practice, this may be taken into account by including the definition

$$\bar{P}_1^1(x) = \sqrt{3}\sqrt{1-x^2} \quad (38)$$

in the base case of the recursion, rather than computing it using the definition of P_ℓ^ℓ. This is the form presented by Holmes and Featherstone [4].

4 Bibliography

[1] M. Abramowitz and I. Stegun. Handbook of mathematical functions. Dover, New York, 1964.

[2] J. A. R. Blais, D. A. Provins, and S. Soofi. Spherical harmonic transforms for discrete multiresolution applications. Journal of Supercomputing, 38(2):173–187, 2006.

[3] A. R. Edmonds. Angular Momentum in Quantum Mechanics. Princeton University Press, Princeton, New Jersey, 1957.

[4] S. A. Holmes and W. E. Featherstone. A unified approach to the Clenshaw summation and the recursive

computation of very high degree and order normalised associated Legendre functions. Journal of Geodesy, 76(5):279–299, 2002.

[5] D. W. Lozier, F. W. J. Olver, and J. M. Smith. Extended-range arithmetic and normalized Legendre polynomials. ACM Transactions on Mathematical Software, 7(1):93–105, 1981.